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DETERMINATION OF THE DEPTH OF THE PLASTIC REGION IN THE PRESSURE OF A FLAT DIE ON A HALF-PLANE

T. P. Pukhnacheva

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Let a flat die of the length ℓ be pressed into a rectilinear boundary without friction so that a pressure distribution q is created under the die. Such a problem was first examined by Prandtl with the assumption that the stresses were continuous everywhere except at the ends of the die and with the use of a constant yield condition (see [1] for example).

Let σ_1 and σ_2 be the principal stresses in the plane (x, y). By means of $2p = \sigma_1 + \sigma_2$ and $2\tau = \sigma_1 - \sigma_2$, any yield condition for an isotropic material can be written in the form $\tau = \tau(p)$.

Using ψ to designate the angle between the first principal direction and the x axis, we express the components of the stress tensor through p, τ, and ψ:

$$\sigma_x = p + \tau\cos 2\psi,\ \sigma_y = p - \tau\cos 2\psi,\quad \sigma_{xy} = \tau\sin 2\psi. \tag{1}$$

Having inserted the equilibrium equation into (1), we obtain the following system of equations:

$$\partial(p + \tau\cos 2\psi)/\partial x + \partial(\tau\sin 2\psi)/\partial y = 0,$$
$$\partial(\tau\sin 2\psi)/\partial x + \partial(p - \tau\cos 2\psi)/\partial y = 0. \tag{2}$$

We assume that $|\tau'| < 1$. Then system (2) is hyperbolic. Two families of characteristic curves and relations along these curves can be written for the system:

$$(\cos 2\psi + \tau')dy = (\sin 2\psi + \sqrt{1-(\tau')^2})dx,$$
$$\psi + \int_{p_0}^{p} \frac{\sqrt{1-(\tau')^2}}{2\tau}\,d\xi = r - \text{const},$$
$$(\cos 2\psi + \tau')dy = (\sin 2\psi - \sqrt{1-(\tau')^2})dx,$$
$$\psi - \int_{p_0}^{p} \frac{\sqrt{1-(\tau')^2}}{2\tau}\,d\xi = s - \text{const}.$$

We will examine the stress field in the plane of the flows (Fig. 1) and the plane of the characteristics. The simplest stress field develops in region ABA_{11}: $p_0 = \tau(p_0) - q$, $\psi = 0$. This region corresponds to the origin of the coordinates in the characteristic plane. The region $A_{11}BA_{21}$ contains a simple centered s-wave, while region $AA_{11}A_{12}$ also contains a simple

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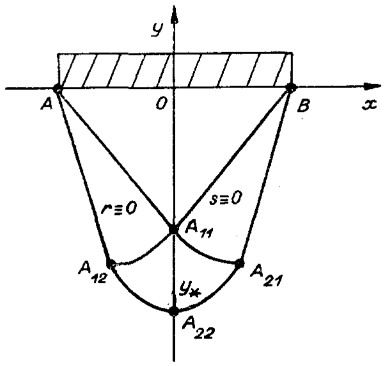

Fig. 1

centered r-wave. Finally, the region $A_{11}A_{12}A_{22}A_{21}$ is the region of interaction of the two simple centered waves. From the right of the line BA_{21} $p_1 = -\tau(p_1)$, $2\psi = \pi$.

It is known that the Prandtl solution is complete, the resulting velocity field is kinematically allowable, and the solution can be continued into rigid zones as far as desired. In the case of an arbitrary yield condition, as before, the Hencky theorem remains valid. By virtue of this theorem, continuation of the extreme right r-characteristic and the extreme left s-characteristic should intersect. We will refer to the point of intersection y_*, on the y axis, as the depth of the plastic zone, and we will explain the method of its determination.

Let us examine the parameters r and s along $A_{11}A_{21}$. We have $s \equiv 0$ or $\psi = \frac{1}{2}\int\limits_{p_0}^{p} \frac{\sqrt{1-(\tau')^2}}{2\tau} \cdot d\xi = \frac{1}{2}\int\limits_{p_0}^{p} \lambda(\xi)\, d\xi$. Thus, along $A_{11}A_{21}$ $r = \int\limits_{p_0}^{p} \lambda(\xi)\, d\xi$. From this, $r_{A_{22}} = r_{A_{21}} = r_* = \int\limits_{p_0}^{p_1} \lambda(\xi)\, d\xi = 2\psi|_{p=p_1} = \pi$.

Being a function of r and s, the function $\tau(p)$ will have the same notation as before. We introduce the new variable $\theta(r - s)$ through the relation $\tau'(r-s) = -\cos\theta(r-s)$. Then the equilibrium equations in the plane of the characteristics in the region $A_{11}A_{12}A_{22}A_{21}$ take the form

$$\frac{\partial x}{\partial r} + \operatorname{tg}\left(\frac{r+s+\theta(r-s)}{2}\right)\frac{\partial y}{\partial r} = 0, \quad \frac{\partial x}{\partial s} + \operatorname{tg}\left(\frac{r+s-\theta(r-s)}{2}\right)\frac{\partial y}{\partial s} = 0. \tag{3}$$

We designate $2\alpha(r, s) = r + s - \theta(r - s)$, $2\beta(r, s) = r + s + \theta(r - s)$.

Since the characteristics $A_{11}A_{12}$ and $A_{11}A_{21}$ and the distributions of the sought variables p and ψ along these curves are known, we can obtain conditions for p and ψ on part of the boundary of the region. Then using the image ih the characteristic plane, we determine the conditions for x and y on segments of the characteristics:

$$\begin{gathered} x(r, 0) = 1 + c\exp\left(\frac{1}{2}\int\limits_0^r \operatorname{ctg}\theta(\xi)\, d\xi\right)\sin\alpha(r, 0)(\sin\theta(r))^{-1/2}, \\ y(0, s) = c\exp\left(\frac{1}{2}\int\limits_0^s \operatorname{ctg}\theta(-\xi)\, d\xi\right)\cos\beta(0, s)(\sin\theta(-s))^{-1/2}, \quad c = \text{const}. \end{gathered} \tag{4}$$

Equations (3) and (4) constitute a Gurs problem. Its solution exists locally and is unique. However, this is no longer obvious for the entire square $[0, \pi][-\pi, 0]$, since the coefficients are discontinuous inside the square along the lines $r + s = \pm(\pi - \theta)$. We will introduce new variables, having subdivided the square into two triangles:

$$\begin{gathered} u = x\operatorname{ctg}\beta + y, \ v = x + y\operatorname{tg}\alpha, \ r \geqslant 0, \ s \geqslant -r, \\ U = x + y\operatorname{tg}\beta, \ V = x\operatorname{ctg}\alpha + y, \ r \geqslant 0, \ s \leqslant -r. \end{gathered} \tag{5}$$

TABLE 1

No.	b	y_*, cm	q, kg/cm²	No.	b	y_*, cm	q, kg/cm²
1	0	—619,8	132,2	1	—0,1	—848,1	173,8
2	0	—619,8	0,514	2	—0,6	—14107,3	6,867
3	0	—247,9	—	3	—0,6	—5642,9	—
1	—0,01	—638,4	135,6	1	—0,6	—14107,3	1765,8
2	—0,3	—1860,1	1,329	2	—0,8	—370162,4	74,06
3	—0,3	—744	—	3	—0,8	—148064,9	—
1	—0,06	—744,5	155,1				
2	—0,4	—3114,2	2,048				
3	—0,4	—1245,7	—				

Remarks: 1) rocks with ℓ = 100 cm, a = 25.714 kg/cm², and values of a and b taken from [2]; 2) clayey soil with ℓ = 100 cm, a = kg/cm², and values of a and b taken from [3]; 3) sandy soil with ℓ = 400 cm, a = 0, and q = 0 [3].

They are connected by a simple linear relation $u = U \operatorname{ctg}\beta, v = V\operatorname{tg}\alpha$, with coefficients which are continuous on the line r + s = 0.

The following relations are satisfied for u, v, U, and V:

$$\frac{\partial v}{\partial s} - \frac{\partial(\operatorname{tg}\alpha)}{\partial s} g_1 (v \operatorname{ctg}\beta - u) = 0, \tag{6}$$

$$\begin{gathered}\frac{\partial u}{\partial r} + \frac{\partial(\operatorname{ctg}\beta)}{\partial r} g_1 (v - u \operatorname{tg}\alpha) = 0, \quad r > 0, \ s \geqslant -r, \\ \frac{\partial V}{\partial s} - \frac{\partial(\operatorname{ctg}\alpha)}{\partial s} g_2 (V \operatorname{tg}\beta - U) = 0, \\ \frac{\partial U}{\partial r} + \frac{\partial(\operatorname{tg}\beta)}{\partial r} g_2 (V - U \operatorname{ctg}\alpha) = 0, \quad r > 0, \quad s \leqslant -r;\end{gathered} \tag{7}$$

$$\begin{gathered}v(r, 0) = 1, \ U(0, s) = -1; \\ u(r, -r) = U(r, -r) \operatorname{ctg}\beta(r, -r), \\ v(r, -r) = V(r, -r) \operatorname{tg}\alpha(r, -r),\end{gathered} \tag{8}$$

where $g_1 = (\operatorname{tg}\alpha \cdot \operatorname{ctg}\beta - 1)^{-1}$; $g_2 = (\operatorname{ctg}\alpha \cdot \operatorname{tg}\beta - 1)^{-1}$.

If we change over to a system of integral equations, it is not hard to show that a solution to problem (6)-(8) for $(r, s) \in [0, \pi] \times [-\pi, 0]$ exists and is unique. Its smoothness depends ultimately on the smoothness of the function $\tau(p)$ and the compatibility conditions at zero. For y, we use (5) to obtain the expression $y = g_1(v \operatorname{ctg}\beta - u)$. The value of $y(\pi, -\pi) = y_*$ will determine the size of the plastic zone along the y axis.

To solve problem (6)-(8), we constructed a stable second-order difference scheme based on the Crank–Nicolson method. In the case when τ is a linear function $\tau(p) = a + bp$, the proposed algorithm is realized in the form of a program in Fortran. The program assigns three constants: a, b, and the length of the die ℓ. The operation of the program gives us the values of y_* and q.

To determine the limiting pressure q, we used the expression $\pi = \int\limits_{p_0}^{p_1} \frac{\sqrt{1-b^2}}{a+b\xi} d\xi = \frac{\sqrt{1-b^2}}{b} \ln \left| \frac{a(1-b)}{(a-bq)(1+b)} \right|$, from which $q = \frac{a}{b}\left[1 - \frac{(1-b)}{(1+b)} \exp\left(-\frac{\pi b}{\sqrt{1-b^2}}\right)\right]$. It is easily seen that $q \to a(\pi + 2)$ as $b \to 0$. This result is consistent with the expression for the limiting pressure in the case of a constant yield condition.

The computing time on a 20 × 20 grid of points is about 30 sec on a BÉSM-6 computer. The results of the calculations are shown in Table 1. Since the initial system (2) is invariant to a tensile transformation, it is sufficient to perform calculations for one value of die length. For the remaining values, the sought depth y_* is determined by simple multiplication.

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